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Description of systematicity intrinsic to the dynamics of complex-system models

Takayuki Nozawa *, Yoshihiro Miyake

Department of Computational Intelligence and Systems Science, G3-822 Miyake-Laboratory, Tokyo Institute of Technology, 4259 Nagatsuta-cho, Midori-ku, Yokohama 226-8502, Japan

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Abstract

To describe *systematicity* in the complex dynamics of deterministic models of various media (worlds), we have proposed a new framework in place of a conventional framework which entails the concept of possible states and the state-set. A significant characteristic of this framework is that it adopts what corresponds to specifications of actual state of limited parts of the world with limited accuracies as the worlds' equally primary elements, instead of introducing statistical ensemble. *Content* of a state of a world is defined not by the position of the state in other possible but unrealized states but by the relations among other realized states. We discuss how the systematicity of the world should be described without its "ground" originating in an external viewpoint, and how differences in some basic features of the dynamics models are distinguished in this framework.

1. Introduction

When we study the complex behavior of various media, such as molecular chemical systems, nervous systems, social systems, and more comprehensive systems (we call these systems *worlds* in general), one purpose is to extract *systematicity* in their dynamics. Here, systematicity means that which defines *systems* (or *individuals* or *unities*) in a world, and characterizes them. Extraction of the systematicity is a problem which is essential to answer the question: "What is a *living system* or an *organism* (e.g., [1])?"

Earlier studies have adopted various, and sometimes conflicting, properties found in patterns or structures of the worlds' dynamics as ones which represent systematicity; for example, integration, homogeneity, stability, complexity, adaptability, hierarchy, and combinations of these. For the quantification of these properties as systematicity, most of the previous studies have depended on characteristics based on a statistical framework, exemplified by information-theoretical characteristics (e.g., [2]). The statistical framework essentially necessitates the concept of a state-set (state space), which is an aggregation of the studied world's *possible but unrealized states*.

This conventional framework characterizes a world from some specific viewpoint, exterior to that world, and has enabled us to study various worlds in a comprehensive way. However, presupposing a priori a structure of knowledge such as the state-set from an external viewpoint may entail risks in deriving systematicity, which is not intrinsic to *the way the world actually is*. For example, when a world changes relatively with respect to a metric presupposed by an external observer, even if the change does not exist from internal viewpoints, the observer "detects" the change, which is not intrinsic to the way the world actually is. Therefore, we would propose the question: "Is it possible to describe systematicity based only on how the world actually is, and not presupposing an a priori structure of knowledge from an external viewpoint?"

* Corresponding author. Tel./fax: +81-45-924-5656.
E-mail address: nozawa@myk.dis.titech.ac.jp (T. Nozawa).

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Following this line of questioning, in this paper we propose a new framework which takes the place of the conventional one for describing systematicity in the dynamics of a world's model (also for which we will use the label "world"). The point of this framework is that its primary elements are *the ways limited parts of the world actually are*. In our framework, the systematicity, intrinsic to the way the world actually is, is described only using relations among these primary elements, independent of an external viewpoint. The state-set is to be reconstructed as a derivative concept.

In Section 2, the constitution of the framework is described, and some choices on fixing the framework are specified. In Section 3, we show how differences in some of the basic features of worlds are discriminated in this framework, and how the choices on the constitution of the framework differentiate the descriptions of the worlds. In Section 4, an example of applying the framework to a simple dynamical model is presented. Finally, in Section 5, topics remaining for future work, and our conclusions, are presented.

2. Constitution of the proposed framework

Observing the dynamics of various worlds only with infinite accuracy, there are equivalent chains of states, with no difference in their systematicity that can be discriminated. This indicates that systematicity is that which can be attributed not to the entire world, but to limited parts of the world. Our framework thus adopts what correspond to how limited parts of the world actually are, as its equally primary elements, instead of presupposing the concept of possible but unrealized states. We call these primary elements SUBJECTSs. The SUBJECTs are the limited parts of the world to which systematicity can be attributed.

In the conventional framework, the *content* of a realized state of the world (with finite or infinite accuracy) is determined by its relative position among the other possible but unrealized states, in the state-set. The "entropy of a state" in statistical information theory [3] is contained within this definition. In contrast, in the framework proposed in this study, the content of a SUBJECT is defined as *the whole of necessity* for the SUBJECT. This should be considered to be what brings the SUBJECT into existence, and is defined by the relations named *connections of necessity* with other SUBJECTs. Consequently, in this framework, the dynamics of a world can be expressed in a directed-graph-like form in which the vertices represent the SUBJECTs, and the edges represent the connections of necessity. The whole of necessity for a SUBJECT is expressed as its partial structure.

We consider that the systematicity of a part of the world is described by the change of its content with the passage of time. Therefore, in this framework, the description of the systematicity in the world's dynamics is given by extracting the time evolution of the whole of necessity for a SUBJECT (and also for an assembly of SUBJECTs). For this extraction, it is necessary to define a sequence of SUBJECTs along which changes of the whole of necessity for the SUBJECT are to be tracked with time. The definition of the sequences to be tracked depends on the choice of viewpoint to specify the self-identity of the SUBJECT.

To sum up, the constitution of the framework proposed in this study is determined by making explicit the following matters:

1. states which correspond to SUBJECTs;
2. the relations which correspond to the connections of necessity;
3. a definition of the whole of necessity for a SUBJECT;
4. a definition of self-identity of a SUBJECT with the passage of time.

In this section, initially the SUBJECT and the connection of necessity are defined, in contrast with the conventional framework (Sections 2.1 and 2.2). Secondly, using these constituent elements of the framework, the whole of necessity for a SUBJECT is defined (Section 2.4). Finally, by defining how to track the time evolution of the whole of necessity for a SUBJECT, the form of description of the systematicity in this framework is shown (Section 2.7).

In this process, we must make a choice about the definition of SUBJECT. The choice determines the difference in what kind of features of the dynamics is to be extracted as a difference in systematicity. Some examples of the choice and features of the dynamics extracted by them are presented in Section 3.

2.1. Definition of SUBJECT

First, in contrast with the conventional framework, we define what concept corresponds to SUBJECT, the primary element of the proposed framework. We stated above that SUBJECTs represent how limited parts of the world actually are, i.e., the specifications of the actual state of the world extended in time from the past until the present, in limited spatio-temporal regions, limited degrees of freedom, and with limited accuracies. Let us examine this in detail.

In the conventional framework based on the state-set (here the concept of states is restricted to the classical one), a state-set, S_t, which represents a set of possible states, is assigned at each time parameterized by $t \in T$ (usually $S_t = S$ for all $t \in T$). The most detailed specification of the way an entire world has actually been, up to the present time, is given as an element $\omega^{t_0} \equiv \{w(t)\}_{t\in T^{t_0}}$ in the direct product $\Omega^{t_0} \equiv \prod_{t\in T^{t_0}} S_t$ of the state-sets. Here, T^{t_0} represents a subset of T, from as far as possible in the past to the present time, represented by $t = t_0$. When a world is defined on $T = \mathbb{R}$, $T^{t_0} = (-\infty, t_0)$. When a world is defined on $T = \mathbb{Z}$, $T^{t_0} = [\dots, t_0 - 2, t_0 - 1, t_0]$.

Initially, the subsets of Ω^{t_0} expressed as

$$L = \prod_{t\in T^{t_0}} L_t, \quad w(t) \in L_t \subsetneqq S_t(t = {}^{\exists}t_1), \qquad L_t = S_t(t \neq t_1), \tag{1}$$

are considered. We label $t_1 \in T^{t_0}$ as *the time referred to* by L, and label L_{t_1} as *the referring set* of L. Here, we define the *actual state of a limited part of the world with a limited precision* – this will hereinafter be abbreviated as ASLP – as a subset of Ω^{t_0}, denoted by the following recursive definition:

1. each subset of Ω^{t_0} with expression (1) referring to any time, t, on T^{t_0} is ASLP;
2. every intersection of ASLPs is ASLP;
3. only subsets of Ω^{t_0} which satisfy either of the conditions 1, 2 are ASLP.

These subsets are denoted by the following expression:

$$L = \prod_{t\in T^{t_0}} L_t, \quad w(t) \in L_t \subsetneqq S_t(t \in {}^{\exists}U), \qquad L_t = S_t(t \notin U). \tag{2}$$

We label $U \subset T^{t_0}$ as *the domain of time referred to* by L, and label L_t, $t \in U$ as *the referring sets* of L. In particular, for "usual" worlds whose time evolution rules derive states at a time only from states at immediately before that time, it is sufficient to consider only the subsets in the form of expression (1) as ASLPs. Note that the subsets of Ω^{t_0} which do not contain ω^{t_0} – the way the entire world actually is – are excluded from the ASLPs.

In the set of ASLPs denoted by expressions (1) or (2), only the ones which satisfy some proper condition have corresponding SUBJECTs. Here, there exists a choice for the condition of the ASLPs to have corresponding SUBJECTs:

What condition is to be imposed on ASLPs to have corresponding SUBJECTs? (3)

The easily handled conditions are ones which reflect structures introduced into S_t or Ω^{t_0} such as topological or metric ones. For example, we restrict the ASLPs which have corresponding SUBJECTs to ones whose referring sets are:

(i) connected with respect to some topology on S_t;
(ii) convex with respect to some linear combination defined on S_t;
(iii) a set of all $s \in S_t$ within a certain distance from $w(t)$ with respect to some metric defined on S_t; or
(iv) restricting the ASLPs to ones which refer to a domain of time that is connected on T^{t_0}, etc.

A guide as to what condition should be imposed relates to the invariance of the systematicity of a world with respect to changes of viewpoint. Now, let (us think that) the systematicity of the world is invariant with respect to some change of viewpoint, e.g., coordinate transformation. Then, to separate only the systematicity intrinsic to the world, the conditions with which the set of all SUBJECTs is dominated by this change of viewpoint should not be imposed. For example, if (we think that) models which are mutually mapped to each other by topologically continuous coordinate transformations represent the same world, and that their systematicity should be identical, then imposing some metrical conditions is not appropriate, because it may cause disagreement in the set of all SUBJECTs among the models.

L denotes a SUBJECT corresponding to an ASLP L. The set of all SUBJECTs is denoted by $\mathscr{S}$. $\mathscr{S}$ also depends on the present time, t_0, although it is not denoted explicitly.

2.2. Definition of connection of necessity

In this section, we determine which relation found between ASLPs is adopted as a connection of necessity between the corresponding SUBJECTs. We define two kinds of connections of necessity between the SUBJECTs; logical connection, CN_L, and causal connection, CN_C. The whole of necessity for a SUBJECT is defined using these connections.

Firstly, it is possible to introduce a semi-order relation between the SUBJECTs being equivalent to the inclusion relation between corresponding ASLPs. For a pair of SUBJECTs, **L** and **K**, if the corresponding ASLPs, L and K, satisfy $L \subset K$, i.e., L is finer than K (since this means that K is a logical necessary condition of L), then we define that a logical connection from **K** to **L** exists. We write this relation as $CN_L(\mathbf{L}, \mathbf{K})$:

$$CN_{\mathrm{L}}(\mathbf{L},\mathbf{K}) \Longleftrightarrow L \subset K. \tag{4}$$

Reversing this correspondence, we can redefine ω^{t_0} as the minimum of $\mathscr{S}$ with respect to CN_{L}, and the concept of state at time t on T^{t_0} with infinite accuracy, as the minimum with respect to CN_{L} of SUBJECTs whose corresponding ASLPs refer to t.

Secondly, we determine stepwise the definition of the causal connection between the SUBJECTs. In this paper, we focus on worlds whose time evolution rules are deterministic. In general, denotation of any state specification is given as a subset of Ω^{t_0}. Hence, the relation of causal concluding defined by the world's time evolution rule possessing the direction of causality is expressed as a relation on the power set of Ω^{t_0}. When $X \subset \Omega^{t_0}$ causally concludes $Y \subset \Omega^{t_0}$, it is expressed as $X\phi Y$.

Earlier, we defined the causal order between SUBJECTs. For a pair of SUBJECTs, **M** and **N**, let U_M and U_N represent the domains of time to which the corresponding ASLPs, M and N, refer, respectively. As **M** *is coexistent with* **N**, it means that $(U_M \subset U_N) \vee (U_M \supset U_N)$ holds. If either of $CN_{\mathrm{L}}(\mathbf{N},\mathbf{M})$ or $CN_{\mathrm{L}}(\mathbf{M},\mathbf{N})$ is satisfied, then **M** is coexistent with **N**. However, even if two SUBJECTs are coexistent, CN_{L} does not necessarily exist between them. If **M** *is prior to* **N**, or if **N** *is posterior to* **M**, then it means that **M** is not coexistent with **N**, and that ${}^{\exists}t_M \in U_M, {}^{\forall}t_N \in U_N(t_M < t_N)$ holds.

When SUBJECTs **M**, **N** and their corresponding ASLPs M, N satisfy the following conditions:

$$(\mathbf{M} \text{ is prior to } \mathbf{N}), \tag{5a}$$

$$^{\forall}Y \subset N\left(^{\exists}Z \subset \Omega^{t_0}(Z\phi Y) \rightarrow {}^{\exists}X \subset M(X\phi Y)\right), \tag{5b}$$

$$^{\forall}X \subset M\left(^{\exists}Z \subset \Omega^{t_0}(X\phi Z) \rightarrow {}^{\exists}Y \subset N(X\phi Y)\right), \tag{5c}$$

we define that **M** causally concludes **N**, writing this relation as $CC(\mathbf{N},\mathbf{M})$.

In general, non-unique SUBJECTs which are not coexistent with each other and which causally conclude the same SUBJECT **N** may exist. Among such SUBJECTs, when **M** is not prior to any others, we say that **M** *is adjacent to* **N**. We let only a SUBJECT that is adjacent to **N** satisfy a necessary condition to have the causal connection, CN_{C}, to **N**.

Non-unique SUBJECTs which are coexistent with each other and causally conclude **N** may also exist in such cases where the world's time evolution rule is non-invertible (see Section 3):

$$\left\{\mathbf{M_i},\ \mathbf{i} \in I \,|\, CC(\mathbf{N},\mathbf{M_i}) \wedge (\mathbf{M_i} \text{ is coexistent with } \mathbf{M_j} \text{ for } \mathbf{i},\mathbf{j} \in I)\right\}. \tag{6}$$

In such SUBJECTs, we allow only one which is maximal with respect to CN_{L}, i.e., which does not receive CN_{L} from any other SUBJECTs in $\{\mathbf{M_i}\}_{\mathbf{i}\in I}$, satisfy a necessary condition to send the causal connection to **N**.

In summing the above conditions, when **M** is the unique SUBJECT which satisfies the following conditions for **N**:

$$CC(\mathbf{N},\mathbf{M}), \tag{7a}$$

$$^{\forall}\mathbf{K} \in \mathscr{S}(CC(\mathbf{N},\mathbf{K}) \rightarrow (\mathbf{M} \text{ is not prior to } \mathbf{K})), \tag{7b}$$

$$^{\forall}\mathbf{K} \in \mathscr{S}((CC(\mathbf{N},\mathbf{K}) \wedge \mathbf{M} \text{ is coexistent with } \mathbf{K}) \rightarrow \neg CN_{\mathrm{L}}(\mathbf{M},\mathbf{K})), \tag{7c}$$

we define that a causal connection from **M** to **N** exists, writing this relation as $CN_{\mathrm{C}}(\mathbf{N},\mathbf{M})$ (see Fig. 1). When such **M** is not determined uniquely, then we allow no SUBJECT to send CN_{C} to **N**.

We elucidate the definition of CN_{C} especially for the usual world whose time evolution rule derives any states at a given time only from states at the time immediately before it. Let a map $\phi^{(t,t')} : S_t \rightarrow S_{t'}$ denote the world's time evolution rule (when the rule is independent of time, it is expressed using one-parameter semigroup $\{\xi^s\}$ as ${}^{\forall}t, t'(t < t')(\phi^{(t,t')} = \xi^{t'-t})$.

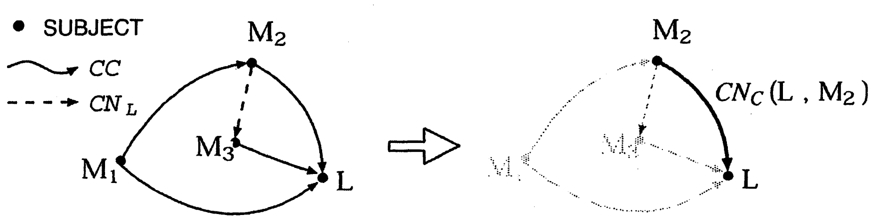

Fig. 1. Schematic representation of the definition of CN_{C}. Let the relations of causal concluding shown on the left-hand side keep between the SUBJECTs. Considering the SUBJECT which sends CN_{C} to **L**, $\mathbf{M}_1$ does not satisfy the condition at (7b). $\mathbf{M}_3$ does not satisfy the condition at (7c). Therefore, only $CN_{\mathrm{C}}(\mathbf{L},\mathbf{M}_2)$ is possible, as on the right-hand side.

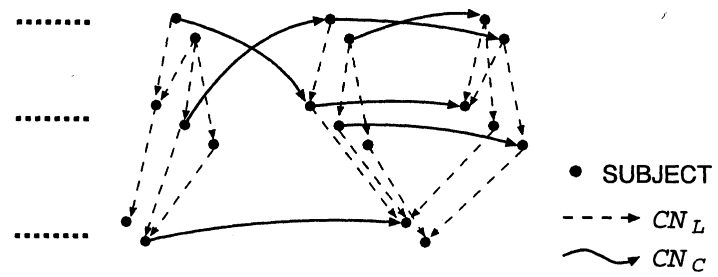

Fig. 2. Schematic representation of the description $(\mathscr{S}, CN_L, CN_C)$. The SUBJECTs are arranged so that the CN_C's turn to the horizontal direction, and the CN_L's turn to the vertical direction. From the definitions of CN_L and CN_C, there may exist many SUBJECTs which send a CN_L to a SUBJECT, but there exists (at most) one SUBJECT which sends a CN_C to a SUBJECT. CN_L satisfies the transitive law.

Here $S_t = S$ for all $t \in T$ and $\xi^s : S \to S$). In these cases, the SUBJECTs to be considered are restricted to ones whose corresponding ASLPs are in the form of expression (1). For such **L**, let $\tau(\mathbf{L})$ denote the time to which its corresponding L refers. Then conditions (5a)–(5c) which define the causal concluding of **N** by **M** are expressed more specifically:

$$\tau(\mathbf{M}) < \tau(\mathbf{N}), \tag{8a}$$

$$^{\forall}y \in N_{\tau(\mathbf{N})}\left({}^{\exists}x \in M_{\tau(\mathbf{M})}\left(\phi^{(\tau(\mathbf{M}),\tau(\mathbf{N}))}(x) = y\right)\right), \tag{8b}$$

$$^{\forall}x \in M_{\tau(\mathbf{M})}\left({}^{\exists}y \in N_{\tau(\mathbf{N})}\left(\phi^{(\tau(\mathbf{M}),\tau(\mathbf{N}))}(x) = y\right)\right). \tag{8c}$$

Here, $M = \{M_t\}$ and $N = \{N_t\}$ are the ASLPs corresponding to **M** and **N**, respectively. And conditions (7a)–(7c) which define the causal connection from **M** to **N** are expressed as

$$CC(\mathbf{N}, \mathbf{M}), \tag{9a}$$

$$^{\forall}\mathbf{K} \in \mathscr{S}(CC(\mathbf{N}, \mathbf{K}) \to \tau(\mathbf{K}) \leq \tau(\mathbf{M})), \tag{9b}$$

$$^{\forall}\mathbf{K} \in \mathscr{S}(CC(\mathbf{N}, \mathbf{K}) \wedge \tau(\mathbf{M}) = \tau(\mathbf{K}) \to \neg CN_L(\mathbf{K}, \mathbf{M})). \tag{9c}$$

We note the *flow of* CN_C. From the definition of CN_C in (7a)–(7c) or in (9a)–(9c), for any **N**, there does not always exist an **M** which is prior to **N**, and satisfies $CN_C(\mathbf{N}, \mathbf{M})$. Likewise, not every **N** sends CN_C to some posterior SUBJECT. When there does not exist a SUBJECT which sends CN_C to a SUBJECT **L**, and **L** sends CN_C to a posterior SUBJECT, it means beginning of a line of flow of CN_C. On the other hand, when there exists a SUBJECT which sends CN_C to a SUBJECT **L**, but **L** does not send CN_C to any posterior SUBJECT, it means a terminating of a line of flow of CN_C. The aspect of such beginning and terminating in the flow of CN_C is an indication of differences in the worlds' time evolution rules, and it depends on the choice of the condition at (3). This point will be illustrated in Section 3.

Fig. 2 describes a world in the form of structure $(\mathscr{S}, CN_L, CN_C)$. Typical changes in the relation between a pair of SUBJECTs are shown in Fig. 3.

2.3. Equivalence relation between worlds

A remarkable feature of our proposed framework is that it describes the world's dynamics in terms of relation between *static* structure specified by logical connection CN_L and *dynamic* structure specified by causal connection CN_C. On this point, the proposed framework resembles the formal model of dynamical system by Louie [4]. As a consequence from this feature, the framework derives an equivalence relation on the set of worlds. In other words, there are classes of worlds, and worlds in a class are not distinguished from the viewpoint of the framework. In this section, we discuss this relation.

To simplify the discussion, we focus on a "usual" world which is defined by a time evolution rule $\{\phi^{(t-\epsilon,t)} : S_{t-\epsilon} \to S_t\}_{t \in T^{t_0}}$ and the way $\omega^{t_0} = \{w(t)\}_{t \in T^{t_0}}$ it has actually been up to t_0. Here, $t - \epsilon$ denotes the time immediately before t (if $T = \mathbb{R}$, then $\epsilon = \mathrm{d}t$, and if $T = \mathbb{Z}$, then $\epsilon = 1$).

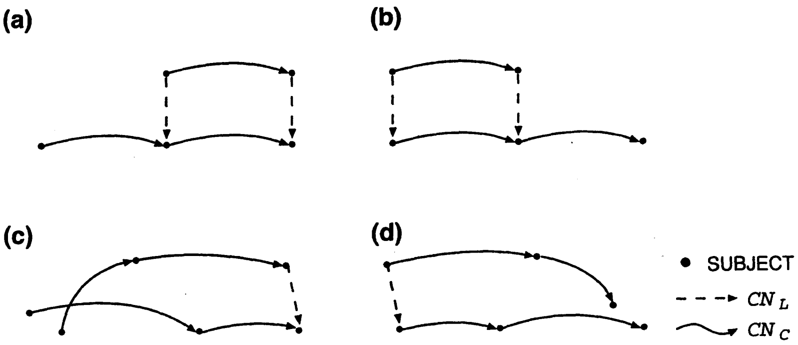

Fig. 3. Typical changes in the relation between a pair of SUBJECTs in the structure $(\mathscr{S}, CN_{\mathrm{L}}, CN_{\mathrm{C}})$. For a pair of sequences of SUBJECTs defined by CN_{C}: (a) CN_{L} becomes held between the two with an arising new line of flow of CN_{C}, (b) CN_{L} becomes broken with the terminating out of a line of CN_{C}, (c) CN_{L} becomes held between the two which are coexistent with each other but CN_{L} did not hold. Both of the lines of flow of CN_{C} last, (d) the inverse of (c). Changes in the form of (c) or (d) are not seen in "usual" worlds. Refer to Section 3.5.

For the state-set S_t at each $t \in T^{t_0}$, we define a family $\mathscr{A}_t$ of its subsets as

$$
\begin{aligned}
&L_A \equiv \prod_{s \in T^{t_0}} L_s, \quad w(t) \in A = L_t \subsetneqq S_t, \qquad L_s = S_s \ (s \neq t), \\
&\mathscr{A}_t \equiv \{A \subset S_t \,|\, \text{ASLP } L_A \text{ has corresponding SUBJECT}\}.
\end{aligned} \tag{10}
$$

$\mathscr{A}_t$ corresponds one-to-one with the total of SUBJECTs whose corresponding ASLPs refer to $t \in T^{t_0}$. From all of the subsets of S_t which contain $w(t)$, the condition at (3) selects out specific ones as elements of $\mathscr{A}_t$ (e.g., only the ones which are connected on S_t are the elements of $\mathscr{A}_t$). The logical connection CN_{L} between SUBJECTs whose corresponding ASLPs refer to $t \in T^{t_0}$ corresponds one-to-one with the inclusion relation $\subset_t$ on $\mathscr{A}_t$. The causal connection CN_{C}, from SUBJECTs whose corresponding ASLPs refer to $t-\epsilon$, to SUBJECTs whose corresponding ASLPs refer to t, corresponds one-to-one with a relation $\Phi^{(t-\epsilon,t)}$ defined as

$$A_{t-\epsilon}\Phi^{(t-\epsilon,t)}A_t \iff \phi^{-(t-\epsilon,t)}(A_t) = A_{t-\epsilon}.$$

$\Phi^{(t-\epsilon,t)}$ is the necessary and sufficient causal concluding relation, defined on $\mathscr{A}_{t-\epsilon} \times \mathscr{A}_t$ (here we ignore the possibility of CN_{C} with skips on T^{t_0}). Getting these results together, the structure $(\mathscr{S}, CN_{\mathrm{L}}, CN_{\mathrm{C}})$ and the structure $(\{\mathscr{A}_t\}, \{\subset_t\}, \{\Phi^{(t-\epsilon,t)}\})_{t \in T^{t_0}}$ are in one-to-one correspondence.

Now, we consider a family $\{\sigma_t\}_{t \in T^{t_0}}$ of bijections from S_t onto itself. The image of $\mathscr{A}_t$ by σ_t,

$$\mathscr{B}_t \equiv \{B \subset S_t \,|\, {}^{\exists}A \in \mathscr{A}_t,\ B = \sigma_t(A)\},$$

is a family of subsets of S_t, with all of its elements containing $v(t) \equiv \sigma_t(w(t))$. For all $t \in T^{t_0}$, let σ_t preserve the structure on S_t brought by the condition at (3) (e.g., σ_t is an isomorphism on S_t). Then, any element B of $\mathscr{B}_t$ satisfies the condition at (3) (e.g., connectivity), and any subset of S_t not belonging to $\mathscr{B}_t$ does not satisfy the condition at (3). Therefore, defining a relation $\Psi^{(t-\epsilon,t)}$ on $\mathscr{B}_{t-\epsilon} \times \mathscr{B}_t$ as

$$
\begin{aligned}
&\psi^{(t-\epsilon,t)} \equiv \sigma_t \circ \phi^{(t-\epsilon,t)} \circ \sigma_{t-\epsilon}^{-1}, \\
&B_{t-\epsilon}\Psi^{(t-\epsilon,t)}B_t \iff \psi^{-(t-\epsilon,t)}(B_t) = B_{t-\epsilon},
\end{aligned}
$$

$\Psi^{(t-\epsilon,t)}$ corresponds one-to-one with $\Phi^{(t-\epsilon,t)}$ on $\mathscr{A}_{t-\epsilon} \times \mathscr{A}_t$.

$$
\begin{tikzcd}
\mathcal{A}_{t-\epsilon} \arrow[r, "\Phi(t-\epsilon,t)"] \arrow[d, "\sigma_{t-\epsilon}"'] & \mathcal{A}_t \arrow[d, "\sigma_t"] \\
\mathcal{B}_{t-\epsilon} \arrow[r, "\Psi(t-\epsilon,t)"'] & \mathcal{B}_t
\end{tikzcd}
$$

And σ_t being bijection, the inclusion relation on $\mathscr{A}_t$ corresponds one-to-one with the inclusion relation on $\mathscr{B}_t$.

Getting these results together, the structure $(\{\mathscr{B}_t\}, \{\subset_t\}, \{\Psi^{(t-\epsilon,t)}\})_{t \in T^{t_0}}$ and the structure $(\{\mathscr{A}_t\}, \{\subset_t\}, \{\Phi^{(t-\epsilon,t)}\})_{t \in T^{t_0}}$ are in one-to-one correspondence. Therefore, the new world, which has $\{\psi^{(t-\epsilon,t)}\}_{t \in T^{t_0}}$ as its time evolution rule and $v^{t_0} \equiv \{\sigma_t(w(t))\}_{t \in T^{t_0}}$ as the way it has actually been up to t_0, is described as the same structure $(\mathscr{S}, CN_{\mathrm{L}}, CN_{\mathrm{C}})$ as the original world, and these two worlds are equivalent from the viewpoint of the proposed framework.

The equivalence relation between worlds is a reflection of the fact that the proposed framework does not presuppose the state-set. The difference in a equivalent class is ascribed to arbitrariness of the external observer's viewpoints.

2.4. Definition of the whole of necessity

In this section, using the causal connection, CN_C, and the logical connection, CN_L, we define the whole of necessity for a SUBJECT, that is the content of the SUBJECT. The point here is that the whole of necessity is defined only by the structure $(\mathscr{S}, CN_L, CN_C)$, and does not inquire of the origins of the SUBJECTs, or the connections of necessity, in the conventional framework based on the state-set.

The complexity of the world's dynamics is reflected in the diversity in the flow of CN_C, especially in the lengths of continuation of CN_C in the causally backward direction. We relate it to the definition of the whole of necessity. First, we define a set $lfCN_C(\mathbf{L})$ of SUBJECTs on a line of flow of CN_C to a SUBJECT $\mathbf{L}$ as follows:

1. if $CN_C(\mathbf{L},\mathbf{M})$, then $\mathbf{M}$ is an element of $lfCN_C(\mathbf{L})$;
2. if $\mathbf{M}$ is an element of $lfCN_C(\mathbf{L})$ and $CN_C(\mathbf{M},\mathbf{N})$, then $\mathbf{N}$ is also an element of $lfCN_C(\mathbf{L})$;
3. only SUBJECTs which satisfy either of the conditions 1, 2 are elements of $lfCN_C(\mathbf{L})$.

Then, we define the whole of necessity for $\mathbf{L}$, which is denoted by $\tilde{\mathbf{L}}$, as a following structure (see Fig. 4):

$$\begin{aligned} &\mathscr{N}(\mathbf{L}) \equiv \{\mathbf{K} \in \mathscr{S} \mid {}^{\forall}\mathbf{M} \in lfCN_C(\mathbf{K}), {}^{\exists}\mathbf{N} \in lfCN_C(\mathbf{L}), CN_L(\mathbf{N},\mathbf{M})\} \\ &\tilde{\mathbf{L}} \equiv (\mathscr{N}(\mathbf{L}), CN_L). \end{aligned} \tag{11}$$

This definition means roughly that in all SUBJECTs which send CN_L to $\mathbf{L}$, only ones whose receiving CN_C continue *not longer* in the causally backward direction than $\mathbf{L}$'s receiving CN_C take part in $\tilde{\mathbf{L}}$. The maximal form of $\tilde{\mathbf{L}}$ is

$$(\{\mathbf{K} \in \mathscr{S} \mid CN_L(\mathbf{L},\mathbf{K})\}, CN_L). \tag{12}$$

The labels $\mathbf{L},\mathbf{M},\mathbf{N},\ldots$ have no contents in themselves. The difference in the content of $\mathbf{L}$, therefore, should be defined only by the difference in the structure of $\tilde{\mathbf{L}}$.

2.5. Meaning of the whole of necessity

The above definition of the whole of necessity was determined by trial and error. In this section, to understand the meaning of the definition, we describe what parts of the world are to have large contents according to the definition.

By definition (11), a SUBJECT $\mathbf{L}$ with the *larger* whole of necessity must satisfy the following conditions:

C1 $\mathbf{L}$ receives CN_L from *more* SUBJECTs;
C2 the CN_C to $\mathbf{L}$ continues *longer* in the causally backward direction;
C3 in SUBJECTs which send CN_L to $\mathbf{L}$, *more* of them receive CN_C which continue not longer in the causally backward direction than the one received by $\mathbf{L}$.

The condition **C1** obviously means that the part which is represented by $\mathbf{L}$ is a *larger* part of the world, which contains more degrees of freedom.

Next, we shall consider the condition **C2**. An ASLP which specifies a limited degrees of freedom is causally concluded by an ASLP which specifies more degrees of freedom. Especially in worlds with infinite degrees of freedom, as we trace backward (and forward as well) the causal concluding, the ASLPs in general specify more and more degrees of freedom, by this "diffusion of causality". Therefore, when some condition which reflects a structure defining *nearness* between states – such as connectivity or convexity – is imposed for the definition of SUBJECT at (3), **C2** requires that in

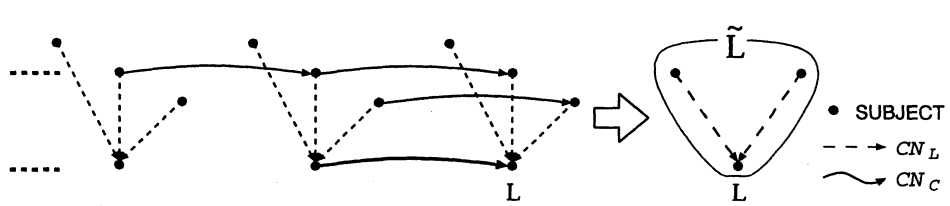

Fig. 4. Schematic representation of the definition of the whole of necessity. When the CN_C's and CN_L between the SUBJECTs are as on the left-hand side, the whole of necessity, $\tilde{\mathbf{L}}$, for $\mathbf{L}$ is defined as on the right-hand side.

tracing backward the causal concluding of **L**, the "diffusion of causality" toward more degrees of freedom should be gentle. In other words, the condition **C2** means that the part of the world which is represented by **L** should be *relatively closed* with respect to causality.

The SUBJECTs which send CN_L to **L** represent sub-parts of the part represented by **L**. Therefore, the condition **C3** means that most of these sub-parts should be *relatively opened* with respect to causality, compared with **L**.

To sum up, by definition (11) of the whole of necessity, a larger part of the world, which is more highly isolated from its environment, and whose internal degrees of freedom interact more strongly, is to have larger content.

2.6. Diversity of the whole of necessity

In this section, we shall mention the diversity of the whole of necessity for SUBJECTs.

In complex worlds, we anticipate the diversity of the whole of necessity, regarding three aspects of the parts of the worlds corresponding to SUBJECTs. That is:

- heterogeneity regarding *which degrees of freedom* the part of the world consisting of;
- non-monotonicity of the whole of necessity regarding *how large* the part of the world being;
- non-monotonicity of the whole of necessity regarding *how detailed precision* the part of the world being specified with.

In Section 2.5, the *size* of the whole of necessity was focused on. From definition (11), however, the whole of necessity can have diversity also with respect to their *complexity*. The diversity of the whole of necessity with respect to their complexity is another aspect which we anticipate for complex worlds.

We shall consider two extreme cases; discussing in the same way as in Section 2.5, when all parts of the world interact equally strongly with the others, CN_C to all SUBJECTs continue uniformly short. Then, by definition (11), for any SUBJECT the whole of necessity is nearly in its maximal form (12), and the diversity does not exist. On the contrary, when all parts of the world are equally isolated from others, CN_C to all SUBJECTs continue uniformly long. Also in this case, the whole of necessity for any SUBJECT is in its maximal form, and the diversity does not exist. As seen from this discussion, for the whole of necessity to be diverse with respect to their complexity, the world should be complex, with the strength of the causal interaction being uneven.

Characterization of the complexity of the whole of necessity is a work of the future. A hopeful idea is to measure it by the size of algorithm needed to describe the whole of necessity $\widetilde{\mathbf{L}} = (\mathscr{N}(\mathbf{L}), CN_L)$ as a directed-graph.

2.7. Description of systematicity

The description of the entire world's dynamics is given as the structure $(\mathscr{S}, CN_L, CN_C)$, and the content of a SUBJECT is given as the whole of necessity for the SUBJECT. Therefore, the systematicity of a part of the world in its dynamics is described as the time evolution of the whole of necessity for a SUBJECT representing the part, in the structure $(\mathscr{S}, CN_L, CN_C)$. For the description, it is necessary to define the sequence of SUBJECTs along which the time evolution of the whole of necessity is to be tracked.

The establishment of the sequence of SUBJECTs is equivalent to the establishment of identity of a SUBJECT with some causally backward SUBJECTs. Standpoints for the establishment of the identity of SUBJECTs are roughly divided into the following two ways:

(Exo) presuppose the identity of SUBJECTs *from an external viewpoint*, independently of the connections of necessity;
(Endo) define the identity of SUBJECTs based only on the connections of necessity.

An example of standpoint **(Exo)** is to consider SUBJECTs whose corresponding ASLPs refer to a specific spatial region, or specific degrees of freedom of the world, with a specific accuracy, to be identical to each other (spatial regions, degrees of freedom, and accuracies are distinguished from an external viewpoint). As examples of standpoint **(Endo)**, one would plainly view SUBJECTs on a line of flow of CN_C as identical SUBJECTs, or it is possible to adopt more complicated settings, where sequences of SUBJECTs which satisfy some maximality with respect to succession of the whole of necessity are sequences of identical SUBJECTs.

The essence of the concept of the state-set in the conventional framework is a presupposition of the identity and grouping of states at different times. Therefore, the conventional framework can be considered to represent standpoint **(Exo)**. On the other hand, **(Endo)** is a new standpoint for examining systematicity, which defines the identity of SUBJECTs based only on the constituent elements of the proposed framework. We believe that **(Exo)** is a shortcut method to treat systematicity from an external viewpoint in an informal way, and for the description of systematicity intrinsic to the world, it is important to pursue some standpoint closer to that of **(Endo)**.

Once the identity of the SUBJECTs is established, it becomes possible to track the time evolution of the whole of necessity. Firstly, for a SUBJECT **L**, we determine the sequence $\{\mathbf{L}', \mathbf{L}'', \ldots\}$ of SUBJECTs identical to it in the causally backward direction. Along this specific sequence of identical SUBJECTs, we can evaluate the following:

- how long the sequence of identical SUBJECTs has lasted;
- along the sequence of SUBJECTs, how the whole of necessity has changed;
- in the change $\ldots, \widetilde{\mathbf{L}''}, \widetilde{\mathbf{L}'}, \widetilde{\mathbf{L}}$ how the ratio of succeeded components has changed. (A succeeded component is also defined by the identity of SUBJECTs.)

Also, by evaluating these matters for an ensemble of SUBJECTs (over the whole of $(\mathscr{S}, CN_{\mathrm{L}}, CN_{\mathrm{C}})$, if possible),

- how sequences of identical SUBJECTs are related to others,

is understood from a viewpoint of looking around the entire world's dynamics. These are all matters that the framework affords for the description of systematicity in the world's dynamics.

3. Basic features of worlds

In this section, we pick up a few cases of how the basic features of a world are reflected in the description of the world's systematicity, and how the choice regarding the definition of SUBJECT at (3) in Section 2.1 influences the description of the dynamics. We limit the discussion here to the difference in the flow of CN_{C}, not mentioning the time evolution of the whole of necessity.

3.1. Invertibility of worlds

Initially, we study the dependence of the flow of CN_{C} on the world's invertibility, imposing no condition at (3). Here, we examine only usual worlds, whose time evolution rules derive states at a time only from states at the time immediately before it. Then, "the world is invertible" means that the time evolution rule, $\phi^{(t,t+s)} : S_t \to S_{t+s}$, is always one-to-one, including both cases of the rule being reversible, i.e., invariant under time reversal, and not invariant under time reversal.

When the world is invertible, for any **N** satisfying $\tau(\mathbf{N}) = t \in T^{t_0}$, **M** with $\tau(\mathbf{M}) = t - \epsilon$ satisfying $CN_{\mathrm{C}}(\mathbf{N}, \mathbf{M})$ is determined uniquely. Conversely, for any **M** satisfying $\tau(\mathbf{M}) = t - \epsilon$, **N**, $\tau(\mathbf{N}) = t$ satisfying $CN_{\mathrm{C}}(\mathbf{N}, \mathbf{M})$ is also determined uniquely. Here, $t - \epsilon$ denotes the time immediately before t. From this, no beginning or terminating exists in the flow of CN_{C}, and all the lines of flow of CN_{C} last from a time to the time immediately after it, without any skips.

In contrast, when the world is non-invertible, meaning that the map $\phi^{(t-\epsilon,t)}$ is multi-valued, for $L_t \subset S_t$, non-unique $M_i \subset S_{t-\epsilon}$ which satisfy $M_i \subsetneq \phi^{(t,t-\epsilon)}(L_t) \wedge \phi^{(t-\epsilon,t)}(M_i) = L_t$ may exist. This corresponds to the existence of non-unique SUBJECTs which are coexistent with each other, and causally conclude the same SUBJECT. In these SUBJECTs, only the one whose corresponding ASLP's referring set is $\phi^{(t,t-\epsilon)}(L_t)$ at $t - \epsilon$ sends CN_{C} to the SUBJECT whose corresponding ASLP's referring set is L_t at t, satisfying the necessary condition at (9c). The other SUBJECTs send CN_{C} to no-one, and hence, many lines of flow of CN_{C} run out.

3.2. Continuity of the dynamics of worlds

When no condition is imposed at (3), the flow of CN_{C} does not depend on the continuity of the world's time evolution rule. However, if some condition reflecting the topological structure of state-set, S_t, such as the connectivity of the ASLPs' referring sets, is imposed, then the flow of CN_{C} depends on the continuity of the time evolution rule.

For a connected ASLP N referring to a time t, the ASLP M which refers to the time $t - \epsilon$ immediately before t and causally concludes N may be disconnected. If this is the case, as the SUBJECT corresponding to M does not exist, then **N** corresponding to N does not receive CN_{C}, at least from the SUBJECTs whose corresponding ASLPs refer to $t - \epsilon$. In particular, if a connected ASLP which causally concludes N does not exist in any previous time, it means that a line of flow of CN_{C} has arisen.

The same discussion holds for the forward direction of causality as well. It means the possibility of terminating in the flow of CN_{C}.

3.3. Metric aspect of the dynamics of worlds

For worlds whose time evolution rules are invertible and continuous, imposing no condition at (3), or imposing some topological conditions, makes no difference to the skeletal structure of the flow of CN_{C}. (Of course, differences are made to the whole of necessity for each SUBJECT, because $\mathscr{S}$ depends significantly on the choice of the condition.) On the other hand, if some condition reflecting the metric structure of the state-set, S_t, such as the convexity of the ASLPs' referring sets is imposed, then some difference among these worlds may appear.

For example, in worlds whose time evolution rules generate chaotic behavior, for a convex ASLP N referring to a time t, the ASLP M which refers to the time $t-\epsilon$ immediately before t and causally concludes N may not be convex from the effect of folding. If this is the case, there exists no SUBJECT corresponding to M, and therefore **N** corresponding to N does not receive CN_{C}, at least from the SUBJECTs whose corresponding ASLPs refer to $t-\epsilon$. In bounded worlds with recurrence, however, a convex ASLP which refers to some time in the past and causally concludes N is expected to exist. In this case, the lines of flow of CN_{C} do not arise or terminate, but may have large and irregular skips on T^{t_0}.

On the other hand, in a world whose time evolution rule maintains the convexity on the state-set, the skeletal structure of the flow of CN_{C} is like the one in the case of imposing no condition at (3).

3.4. Worlds with spatial extent

In the examples given above, their state-sets do not (explicitly) have spatial extent. However, in many worlds, the state-set, S_t, is decomposed into space D and set $Q_{d,t}$ of local states at every position $d \in D$ as $S_t = \prod_{d\in D} Q_{d,t}$ ($S = Q^D$ if $Q_{d,t} = Q$ for all $d \in D$ and $t \in T^{t_0}$). Then, we can classify the conditions at (3) into ones concerning structures of the set of local states, Q, and ones concerning structures of the space, D. Here, we discuss the dependence of the flow of CN_{C} on the conditions reflecting structures of the space, D, imposed at (3).

When the state-set is expressed as $S_t = \prod_{d\in D} Q_{d,t}$, the referring sets of an ASLP L in the form of expression (2) are decomposed in the following form:

$$L_t = L_t^{(V)} \times \prod_{u\in D-V} Q_{u,t}, \quad L_t^{(V)} \subsetneqq \prod_{v\in V} Q_{v,t}. \tag{13}$$

We call V *the domain of space referred* by L. The first term of the right-hand side represents specification of local states of the domain of space referred to by L, and the second term represents the domain of space which is not referred to by L.

For example, if our definition states that only the ASLPs which refer to the domains of space connected with respect to a topology on D – here we call them spatially connected ASLPs – have corresponding SUBJECTs, then it leads to a parallel discussion to the one above concerning the continuity of the world's dynamics. That is, when the time evolution rule contains only actions through medium, keeping continuity on the space, then any SUBJECT corresponding to a spatially connected ASLP receives CN_{C} from a SUBJECT corresponding also to a spatially connected ASLP. Therefore, imposing topological conditions at the (3) does not affect the skeletal structure of the flow of CN_{C}. In contrast, when the time evolution rule contains some action at distance between disconnected spatial regions, for a spatially connected ASLP, the spatially connected ASLP which causally concludes it (is causally concluded by it) may not be determined uniquely. This implies a beginning (terminating) line of flow of CN_{C}.

3.5. More extraordinary worlds

In all the cases discussed above, the worlds' time evolution rules derive any states at a time only from states at the time immediately before it. For this reason, as stated in Section 2.1, it is sufficient to consider only the SUBJECTs whose corresponding ASLPs are in the form of expression (1). In these cases, as long as the time evolution rules are deterministic, the CN_{L} relation between two sequences of SUBJECTs defined by CN_{C} does not change without beginning or terminating of either of the lines of flow of CN_{C}. Therefore, the changes shown in Fig. 3(c) and (d) do not occur.

As worlds where such changes in the CN_{L} relation occur, ones whose time evolution rule have a non-uniform delay, or ones where the $\mathscr{S}$ depends on each present time may be possible. The pursuit of such worlds with extraordinary features seems to lead us especially to the study of concepts of non-classical states. However, the possibility of physically meaningful worlds which generate such situation is a question remaining for future investigation.

4. Application to second-order ECA

In this section, an application of the proposed framework to very simple dynamical models as worlds is shown. The models studied here are second-order elementary cellular automata. The definitions and discussions in Sections 2 and 3 are concretized through this example.

4.1. Formulation of second-order ECA

Cellular automata (CA) are abstract dynamical models, each site i on discrete space (lattice) I is associated a local state variable q_i ranging over a finite set $\{0, 1, \ldots, k-1\} \equiv Q$, and they are updated synchronously at every discrete time step $t \in \mathbb{Z}$.

Dynamics of a first-order CA is defined by the following *local evolution rule*, which determines the local state of a site i at time $t+1$ from local states of neighbors of the site i at t:

$$q_i^{t+1} = f q_{i+X}^t. \tag{14}$$

X is a finite set called *neighborhood*, which is applied to a site i yielding i's neighbors $i + X \equiv \{i + x | x \in X\}$. f is a function called *rule table*, which takes $|X|$ number of local states as arguments, and returns a local state. The local evolution rule induces a *global evolution rule* ϕ, which is applied to a *configuration* $s(t) \equiv (q_i^t)_{i \in I} \in Q^I \equiv S$ of the CA:

$$\phi : s(t) \mapsto s(t+1), \quad t \in \mathbb{Z}.$$

In general, the global evolution rule of first-order CA is non-invertible. That is, different configurations $s(t), r(t) \in S$ may yield the same $s(t+1)$.

Now, let us consider cases where the lattice I is one-dimensional ($I = \mathbb{Z}$), the local state variables can take one of $k = 2$ values ($q_i \in \{0, 1\} = Q$), and the neighborhood $X = \{-1, 0, 1\}$. Then, the local evolution rules of these first-order CA are represented as

$$q_i^{t+1} = f(q_{i-1}^t, q_i^t, q_{i+1}^t). \tag{15}$$

These specifically simple first-order CA are called elementary cellular automata (ECA), and their dynamics has been studied thoroughly in various aspects [5]. A rule table f of first-order CA takes three binary variables as arguments, and returns one binary variable. So there are $2^{2^3} = 256$ possible rules. We can describe a rule table f by a set of "possible arguments/returned value" as

$$\begin{array}{l} 111/c_7,\ 110/c_6,\ 101/c_5,\ 100/c_4 \\ 011/c_3,\ 010/c_2,\ 001/c_1,\ 000/c_0 \end{array} \quad (c_j \in \{0, 1\}) \tag{16}$$

and assign a *rule number* $R \equiv \sum_{n=0}^{7} c_n 2^n$ uniquely to each possible rule table [6]. Rules which satisfy $c_1 = c_4$ and $c_3 = c_6$ are called *symmetrical rules*. There are $2^{8-2} = 64$ possible symmetrical rules.

Now, we show the formulation of second-order CA. Lattice, time, local states, neighborhood and neighbors are defined in the same way as in first-order CA. The local evolution rules of second-order CA are defined as

$$q_i^{t+1} = f q_{i+X}^t - q_i^{t-1} \bmod k. \tag{17}$$

For a determination of a local state at $t+1$, second-order CA require not only local states of its neighbors at t, but also the local state at $t-1$. From the above definition, when we define the configuration at t as $s(t) \equiv ((q_i^{t-1})_{i \in I}, (q_i^t)_{i \in I}) \in Q^I \times Q^I \equiv S$, the global evolution rule

$$\phi : s(t) \mapsto s(t+1), \quad t \in \mathbb{Z},$$

always becomes invertible [7]. This is proved by the fact that the *inverse local rule* derived from expression (17)

$$q_i^{t-1} = f q_{i+X}^t - q_i^{t+1} \bmod k, \tag{18}$$

determines $(q_i^{t-1})_{i \in I}$ uniquely from $(q_i^t)_{i \in I}$ and $(q_i^{t+1})_{i \in I}$. The existence of the inverse local rule makes easy the application of our proposed framework to second-order CA.

Corresponding to definition (15) of first-order ECA, second-order ECA are defined

$$q_i^{t+1} = f(q_{i-1}^t, q_i^t, q_{i+1}^t) - q_i^{t-1} \bmod 2. \tag{19}$$

The evolution rule of second-order ECA is also specified by the rule number of f. Space–time patterns generated by second-order ECA with some rules are shown in Fig. 5.

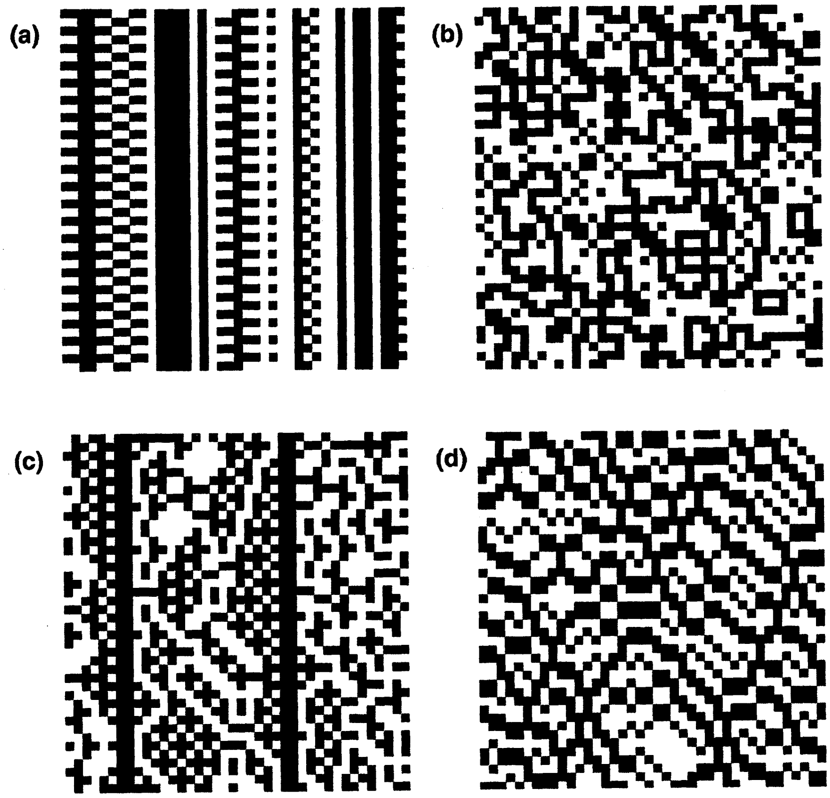

Fig. 5. Space–time patterns of some second-ECA rules with time flowing downwards. The rules act upon randomly chosen initial configurations on a lattice with $|I| = 40$ sites with periodic boundary condition. Black and white correspond to local states 0 and 1, respectively. (a) $R = 0$, (b) $R = 150$, (c) $R = 104$, (d) $R = 95$.

4.2. SUBJECT

Here, the definition of SUBJECT in Section 2.1 is applied to second-order ECA.

ASLPs are represented as denotations of *partial state specifications*. Fundamental element of the partial state specification is a *pointwise state specification*

$$q_j^{t_1} = c(c \in Q) \tag{20}$$

of a local state of a site $j \in I$ at a time $t_1 \in T^{t_0}$. This pointwise state specification has a subset

$$\{(q_j^{t_1} =)c\} \times \prod_{t\in T^{t_0}, i\in I, (t,i)=(t_1,j)} Q \tag{21}$$

of $\Omega^{t_0} \equiv \prod_{t\in T^{t_0}} \prod_{i\in I} Q$ as its denotation.

Let $\omega^{t_0} \equiv (q_i^t = w_i^t)_{i\in I}^{t\in T^{t_0}}$ the way the world has actually been, up to the present time t_0. When a pointwise state specification (20) satisfies $c = w_j^{t_1}$, its denotation (21) contains ω^{t_0}. In this case, we say that this pointwise state specification is *true*. Any ASLP in the form of expression (1) is represented as the denotation of one of the following partial state specifications:

(1) pointwise state specifications which are true;
(2) partial state specifications obtained as a conjunction ($\wedge$) of pointwise state specifications, all of which are true;
(3) partial state specifications obtained as a disjunction ($\vee$) of pointwise state specifications, at least one of which is true;
(4) partial state specifications obtained by connecting ones which satisfy one of 1–3 arbitrarily with $\wedge$ and $\vee$.

Now, it is necessary to determine the condition at (3) in Section 2.1 for ASLPs to have corresponding SUBJECTs. For simplicity, we define that only ASLPs which are denotations of partial state specifications in the following form:

$$\left(q_i^t \cdots q_{i+l}^t; q_i^{t-1} \cdots q_{i+l}^{t-1}\right) = \left(w_i^t \cdots w_{i+l}^t; w_i^{t-1} \cdots w_{i+l}^{t-1}\right) \quad (l \geqslant 0,\ t \in T^{t_0}), \tag{22}$$

which are obtained as a conjunction of true pointwise state specifications of an interval $i, \dots, i+l$ $(l \geqslant 0)$ on the lattice I at a pair of consecutive times $t, t-1$, have corresponding SUBJECTs (Fig. 6(a)).

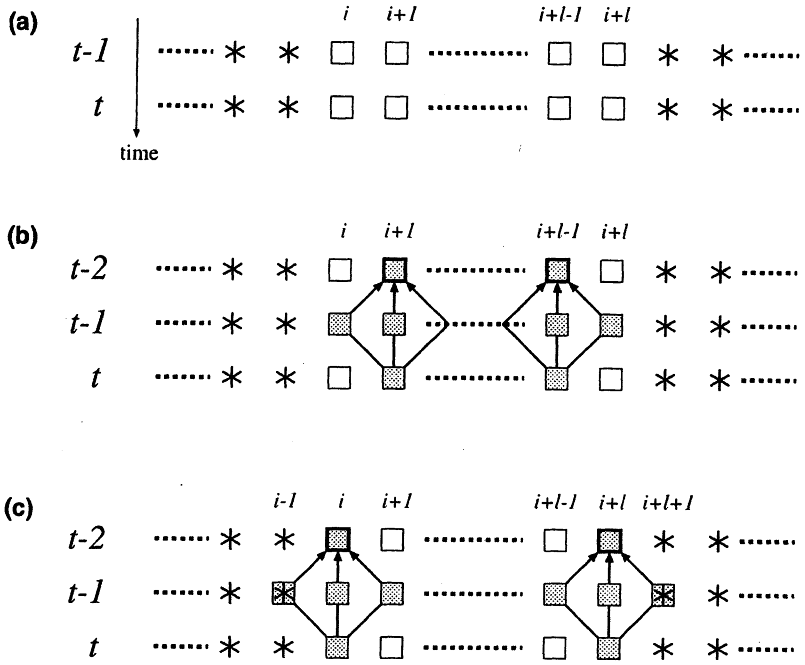

Fig. 6. Partial state specifications which have correspondent SUBJECTs, and the determination of the causal concluding relation between them are shown. The rectangles (□) represent points where the true local states are specified, and the asterisks (*) represent points where the local states are not specified, on the space–time. (a) We define that only ASLPs as denotations of the partial state specifications in this form have corresponding SUBJECTs. In determining the partial state specification at $(t-1, t-2)$ which causally concludes the partial state specification at $(t, t-1)$, (b) local states of sites $j \in \{i+1, \dots, i+l-1\}$ at $t-2$ are determined uniquely by the local states of sites $j \in \{i+1, \dots, i+l-1\}$ at t and the local states of sites $k \in \{i, \dots, i+l\}$ at $t-1$. On the other hand, (c) to determine the local states of sites i and $i+l$ at $t-2$, all possible local states of sites $i-1$ and $i+l+1$ at $t-1$ must be considered, respectively.

This condition for ASLPs to have corresponding SUBJECTs is very restrictive, and we see it as follows. The *Hamming distance H* between two configurations $s(t), r(t) \in S$, $s(t) \equiv ((q_i^t)_{i\in I}; (q_i^{t-1})_{i\in I})$ and $r(t) \equiv ((p_i^t)_{i\in I}; (p_i^{t-1})_{i\in I})$ at t, can be defined as

$$H(s(t), r(t)) \equiv \sum_{i\in I} |q_i^t - p_i^t| + \sum_{i\in I} |q_i^{t-1} - p_i^{t-1}|. \tag{23}$$

In terms of the Hamming distance, a connectivity of subsets of S can be defined; when for any pair of elements x, y of a subset A of S there exists a sequence of elements $\{a_i \in A\}_{i\in\{1,\dots,N\}}$ of A (N may be infinity if A is an infinite set) and they satisfy

$$H(x, a_1) = 1,$$

$$\vdots$$

$$H(a_i, a_{i+1}) = 1 \quad (i \in \{1, \dots, N-1\}),$$

$$\vdots$$

$$H(a_N, y) = 1.$$

Let A be connected on S. It is evident that all ASLPs as denotations of the partial state specifications in the form of expression (22) are connected in the term of this definition of connectivity.

4.3. Connection of necessity

4.3.1. Logical connection CN_L

From the definition (4) of logical connection CN_L in Section 2.2, existence of CN_L between two SUBJECTs is determined by the inclusion relation between the spatial intervals to which the corresponding two partial state specifications are referring.

We shall consider a partial state specification in the form of expression (22), which specifies the local states of an interval $[i, \dots, i+l]$ of $l+1$ in width, at $(t, t-1)$. This partial state specification includes two state specifications of

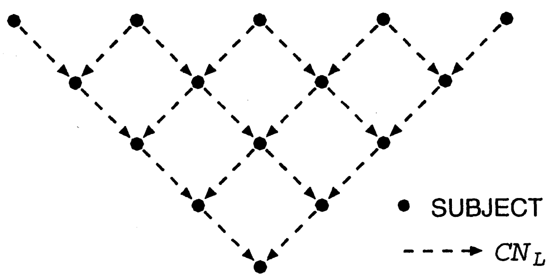

Fig. 7. For a SUBJECT corresponding to a partial state specification in the form of expression (22) of an interval of $l+1=5$ in width (represented by the bottom vertex), the total of SUBJECTs which send CN_{L} to it is shown. CN_{L} which can be obtained by the transitive law are omitted.

intervals of l in width, $[i,\dots,i+l-1]$ and $[i+1,\dots,i+l]$, at the same pair of times $(t,t-1)$. It also includes three partial state specifications of intervals of $l-1$ in width, $[i,\dots,i+l-2]$, $[i+1,\dots,i+l-1]$, and $[i+2,\dots,i+l]$, and so forth. And finally, it includes $l+1$ pointwise state specifications.

From this inclusion relation between partial state specifications, for a SUBJECT which corresponds to the partial state specification of an interval of $l+1$ in width, the total of SUBJECTs which send CN_{L} to it is arranged in a pyramidal form with $l+1$ layers (Fig. 7).

4.3.2. Causal connection CN_{C}

Second-order ECA are "usual" worlds, the entire state of which at time t is specified by a configuration $s(t)=((q_i^t)_{i\in I};(q_i^{t-1})_{i\in I})$, and $s(t)$ is determined only from $s(t-1)$. Therefore, the causal connections between SUBJECTs are determined according to definitions (8a)–(8c) and (9a)–(9c) in Section 2.2.

Firstly, we show how the causal concluding relations between partial state specifications are determined by the inverse of the local evolution rule (19). Here we focus on the partial state specifications in the form of expression (22) with $l\geqslant 2$, at $(t,t-1)$. The cases of $l=0$ and 1 can be discussed in the same way.

Local states $q_{i+1}^{t-2}\cdots q_{i+l-1}^{t-2}$ are determined uniquely from $q_{i+1}^{t}\cdots q_{i+l-1}^{t}$ and $q_i^{t-1}\cdots q_{i+l}^{t-2}$, by the inverse of the local evolution rule (19) (Fig. 6(b)). On the other hand, to determine a local state q_i^{t-2}, each of the possible values in $\{0,1\}$ of q_{i-1}^{t-1}, which the original partial state specification does not specify, must be considered. In the same way, to determine q_{i+l}^{t-2}, each of the possible values in $\{0,1\}$ of q_{i+l+1}^{t-1} must be considered (Fig. 6(c)). Accordingly, the partial state specification at $(t-1,t-2)$ which causally concludes, necessarily and sufficiently, the partial state specification in the form of expression (22) at $(t,t-1)$ is expressed as

$$\begin{aligned}\left(q_{i-1}^{t-1}\cdots q_{i+l+1}^{t-1};q_i^{t-2}\cdots q_{i+l}^{t-2}\right) &= \left(0\,w_i^{t-1}\cdots w_{i+l}^{t-1}\,0;\,l_0\,w_{i+1}^{t-2}\cdots w_{i+l-1}^{t-1}\,r_0\right)\vee\left(0\,w_i^{t-1}\cdots w_{i+l}^{t-1}\,1;\,l_0\,w_{i+1}^{t-2}\cdots w_{i+l-1}^{t-1}\,r_1\right)\\ &\quad\vee\left(1\,w_i^{t-1}\cdots w_{i+l}^{t-1}\,0;\,l_1\,w_{i+1}^{t-2}\cdots w_{i+l-1}^{t-1}\,r_0\right)\vee\left(1\,w_i^{t-1}\cdots w_{i+l}^{t-1}\,1;\,l_1\,w_{i+1}^{t-2}\cdots w_{i+l-1}^{t-1}\,r_1\right).\end{aligned}$$

Here

$$\begin{aligned}l_0 &= f\left(0,w_i^{t-1},w_{i+1}^{t-1}\right)-w_i^t \bmod 2,\\ l_1 &= f\left(1,w_i^{t-1},w_{i+1}^{t-1}\right)-w_i^t \bmod 2,\\ r_0 &= f\left(w_{i+l-1}^{t-1},w_{i+l}^{t-1},0\right)-w_{i+l}^t \bmod 2,\\ r_1 &= f\left(w_{i+l-1}^{t-1},w_{i+l}^{t-1},1\right)-w_{i+l}^t \bmod 2.\end{aligned}$$

When $l_0=l_1$ and $r_0=r_1$ are satisfied, this partial state specification at $(t-1,t-2)$ agrees with the form of expression (22). Then it has corresponding SUBJECT, which sends causal connection CN_{C} to the SUBJECT which corresponds to the original partial state specification at $(t,t-1)$. Contrary to this, when $l_0\neq l_1$ or $r_0\neq r_1$, the partial state specification does not agree with the form of expression (22). And besides, by tracing backward the causal concluding of it further, one can never get to a partial state specification with the form of expression (22). Therefore in this case, the SUBJECT corresponding to the original partial state specification at $(t,t-1)$ does not receive CN_{C}.

Consequently, the condition that only the partial state specifications in the form of expression (22) have their corresponding SUBJECTs is proved to be a definition which make CN_{C} apply only when the "diffusion of causality" toward wider spatial domain does not occur. This is a special example of the discussion in Section 2.5.

Based on the above definitions, second-order ECA with symmetrical rules can be divided into four classes (A) to (D), according to the differences in the flow of CN_{C}, as in Table 1. In second-order ECA with class (A) rules, all SUBJECTs

Table 1
Classification of second-order ECA with symmetrical rules, according to the difference in the flow of CN_C

Class	Len	Inf	Rules
(A)	—	∃	$R = 0, 51, 204, 255$
(B)	—	—	$R = 90, 105, 150, 165$
(C)	∃	∃	$R = 1, 4, 5, 18, 19, 22, 23, 32, 33, 36, 37, 50, 54,$ $55, 72, 73, 76, 104, 108, 126, 127, 128, 129, 132, 133,$ $146, 147, 160, 161, 164, 179, 200, 201, 205, 219, 223, 232, 236, 251, 254$
(D)	∃	—	$R = 77, 91, 94, 95, 109, 122, 123, 151, 178, 182, 183, 218, 222, 233, 237, 250$

The column labelled "Len" gives the existence of variety in the lengths of continuation of CN_C for different SUBJECTs. The column labelled "Inf" gives the existence of the lines of flow of CN_C which continue infinitely. "∃" and "—" represent existence and non-existence, respectively.

receive CN_C, and hence no beginning or terminating exists in the flow of CN_C. An example is $R = 0$, with its typical dynamics shown in Fig. 5(a). In second-order ECA with class (B) rules, on the other hand, no SUBJECTs receive CN_C, and hence the flow of CN_C does not exist. An example is $R = 150$, with its typical dynamics shown in Fig. 5(b). With class (C) or (D) rules, the presence of CN_C depends on each SUBJECT, and there is a variety in the flow of CN_C. In second-order ECA with class (C) rules, some lines of flow of CN_C may continue infinitely. Such lines of flow of CN_C correspond to periodic "wall" patterns which divide the spatial lattice I into isolated sub-regions. The "wall" pattern is exemplified in Fig. 5(c), which is generated with a class (C) rule $R = 104$. Time evolutions of spatial domains between two "walls" become inevitably periodic, although the period may be considerably long. In second-order ECA with class

Table 2
With rule $R = 104$ which belongs to class (C), tracing backward the causal concluding relations for possible partial state specifications in the form of expression (22), of intervals of $l + 1 = 3$ in width, are shown

t	000	000	000	000	000	000	000	000
$t-1$	000	100	010	001	110	101	011	111
$t-2$	000	?00	?0?	00?	?1?	?1?	11?	?0?
$t-3$	**	–	–	–	–	–	–	–
t	100	100	100	100	100	100	100	100
$t-1$	000	100	010	001	110	101	011	111
$t-2$	100	?00	?0?	10?	?1?	?1?	?1?	?0?
$t-3$	?00	–	–	–	–	–	–	–
t	010	010	010	010	010	010	010	010
$t-1$	000	100	010	001	110	101	011	111
$t-2$	010	?10	?1?	01?	?0?	?0?	?0?	?1?
$t-3$	?0?	–	–	–	–	–	–	–
t	101	101	101	101	101	101	101	101
$t-1$	000	100	010	001	110	101	011	111
$t-2$	101	?01	?0?	10?	?1?	?1?	?1?	?0?
$t-3$	?1?	–	–	–	–	–	–	–
t	110	110	110	110	110	110	110	110
$t-1$	000	100	010	001	110	101	011	111
$t-2$	110	?10	?1?	11?	?0?	?0?	?0?	?1?
$t-3$	?1?	–	–	–	–	–	–	–
t	111	111	111	111	111	111	111	111
$t-1$	000	100	010	001	110	101	011	111
$t-2$	111	?11	?1?	11?	?0?	?0?	?0?	?1?
$t-3$	?0?	–	–	–	–	–	–	–

Some of the partial state specifications which can be obtained by reflection are omitted. "?" means that the local state at the time depends on the local states outside the interval, and hence the line of flow of CN_C begins at the time. "**" represents the same pattern is periodically repeated, and it means the existence of "wall" pattern.

Table 3
With rule $R = 95$ which belongs to class (D), tracing backward the causal concluding relations for possible partial state specifications in the form of expression (22), of intervals of $l + 1 = 3$ in width, are shown

t	000	000	000	000	000	000	000	000
$t-1$	000	100	010	001	110	101	011	111
$t-2$	111	111	?1?	111	?1?	101	?1?	?0?
$t-3$	?0?	?0?	–	?0?	–	000	–	–
$t-4$	–	–	–	–	–	010	–	–
$t-5$	–	–	–	–	–	?1?	–	–
t	100	100	100	100	100	100	100	100
$t-1$	000	100	010	001	110	101	011	111
$t-2$	010	011	?1?	011	?1?	011	?1?	?0?
$t-3$	?1?	?1?	–	?1?	–	?1?	–	–
t	010	010	010	010	010	010	010	010
$t-1$	000	100	010	001	110	101	011	111
$t-2$	101	101	?0?	101	?0?	101	?0?	?1?
$t-3$	101	001	–	100	–	000	–	–
$t-4$	000	010	–	010	–	010	–	–
$t-5$	010	?1?	–	?1?	–	?1?	–	–
$t-6$	?1?	–	–	–	–	–	–	–
t	101	101	101	101	101	101	101	101
$t-1$	000	100	010	001	110	101	011	111
$t-2$	010	010	?1?	010	?1?	000	?1?	?0?
$t-3$	?1?	?1?	–	?1?	–	010	–	–
$t-4$	–	–	–	–	–	?1?	–	–
t	110	110	110	110	110	110	110	110
$t-1$	000	100	010	001	110	101	011	111
$t-2$	011	001	?0?	001	?0?	011	?0?	?1?
$t-3$	?1?	011	–	110	–	?1?	–	–
$t-4$	–	?1?	–	?1?	–	–	–	–
t	111	111	111	111	111	111	111	111
$t-1$	000	100	010	001	110	101	011	111
$t-2$	000	000	?0?	000	?0?	010	?0?	?1?
$t-3$	111	011	–	110	–	?1?	–	–
$t-4$	?0?	?1?	–	?1?	–	–	–	–

The notation is the same as that in Table 2. "**" indicating the existence of "wall" patterns does not appear.

(D) rules, on the other hand, no "wall" patterns exist, and the time evolutions become aperiodic in general. An example is $R = 95$, with its typical dynamics shown in Fig. 5(d).

For possible partial state specifications in the form of expression (22) with $l = 2$, how long the causal concluding relations are traced backward, with $R = 104$ and $R = 95$, are shown in Tables 2 and 3, respectively. They indicate a part of the variety in the flow of CN_C.

4.4. The whole of necessity

From definition (11) in Section 2.4, the whole of necessity for a SUBJECT **L** which corresponds to a partial state specification in the form of expression (22) is determined by the following procedure; firstly, it should be obtained how long the CN_C received by **L** continues in the causally backward direction. Secondly, also for all SUBJECTs which send CN_L to **L**, how long the CN_C received by each of them continues in the causally backward direction should be obtained. Then, in all these SUBJECTs, only ones whose receiving CN_C continue not longer than **L**'s receiving CN_C take part in **L** (Fig. 8). Therefore, the whole of necessity $\tilde{\mathbf{L}}$ for **L** is, in general, a part of the structure (12) shown in Fig. 7.

According to this procedure, in second-order ECA with class (A) or (B) rules, for all SUBJECTs, the whole of necessity are in their maximal forms (Fig. 8(a) and (b)). With class (C) rules, SUBJECTs which correspond to the partial state specifications of regions between two "wall" patterns have the maximal whole of necessity, which does not change

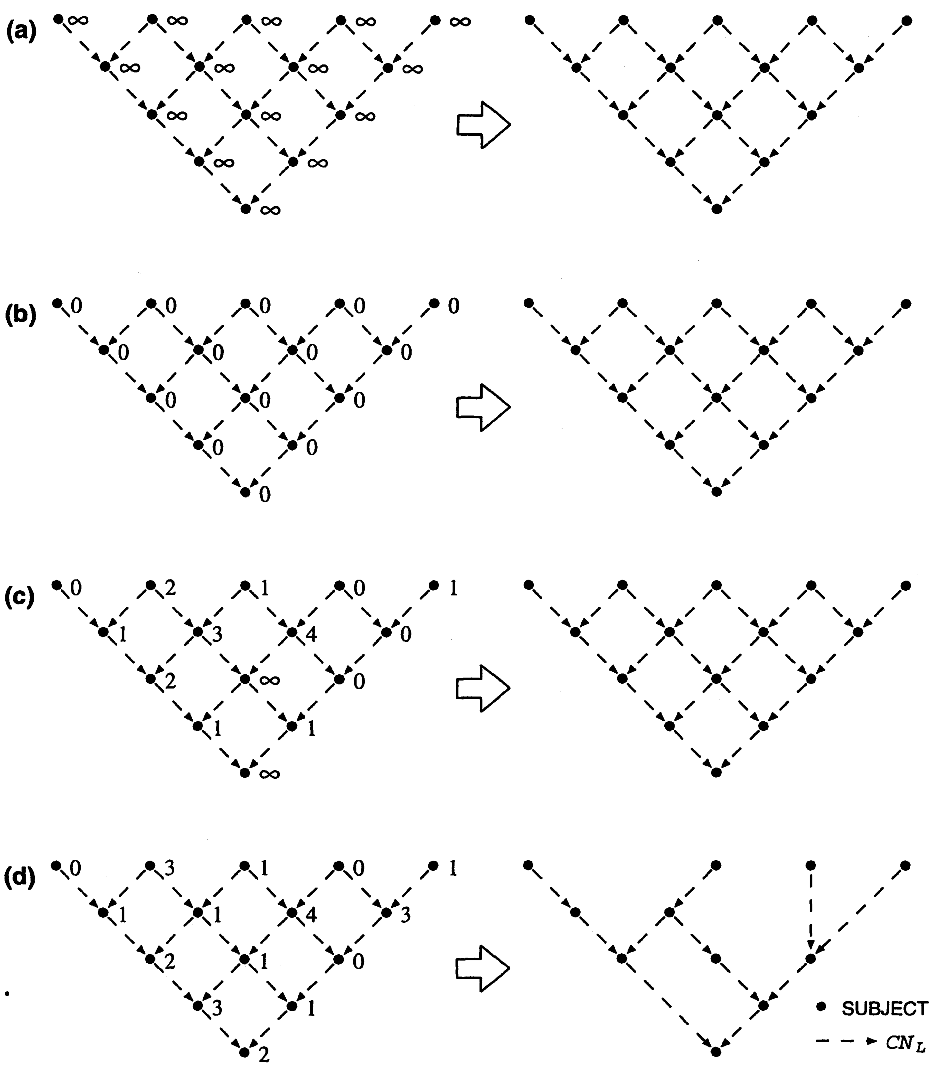

Fig. 8. The determination of the whole of necessity. For SUBJECTs represented by the vertices, let the numbers attached to each of them on the left-hand side represent the lengths of continuation of CN_C to them in the causally backward direction. Then the whole of necessity for the SUBJECT represented by the bottom vertex is determined as on the right-hand side. (a) Cases where all SUBJECTs receive CN_C which continue infinitely. (b) Cases where no SUBJECTs receive CN_C. (c) The SUBJECT receives CN_C which continues infinitely. (d) An example in general circumstances with class (C) or (D). CN_L which can be obtained by the transitive law are omitted.

in the time evolution (Fig. 8(c)). Here we give some examples about the whole of necessity in a dynamics of second-order ECA with a class (D) rule $R = 95$, which will concretize the discussions in Sections 2.6 and 2.7.

What are shown in Fig. 9 are the whole of necessity for some SUBJECTs, whose corresponding partial specifications are about different spatial regions at different times. The differences between (a) and (b), and (c) and (d) illustrate the heterogeneity of the whole of necessity regarding *which degrees of freedom* the part of the world consisting of. In addition, the partial state specification of (c) includes that of (a), and the partial state specification of (d) includes that of (b), respectively. Therefore, the difference between "(a)–(c)" and "(b)–(d)" illustrates the non-monotonicity of the whole of necessity regarding *how large* the part of the world being. [1]

In Fig. 10, some examples of the time evolution of the whole of necessity are shown. In the succession from (a″) to (a′) to (a), the SUBJECTs are on the same line of flow of CN_C. From (b″) to (b′) to (b), on the other hand, CN_C does not hold between the SUBJECTs. Therefore, when two SUBJECTs connected by CN_C are defined to be identical, from the standpoint **(endo)** in Section 2.7, (a)–(a″) shows the time evolution of the whole of necessity for identical SUBJECTs, but (b)–(b″) shows merely the comparison of the whole of necessity for different SUBJECTs.

[1] The non-monotonicity of the whole of necessity regarding *how detailed precision* the part of the world being specified with is omitted from the discussion here, by the definition of SUBJECT in Section 4.2.

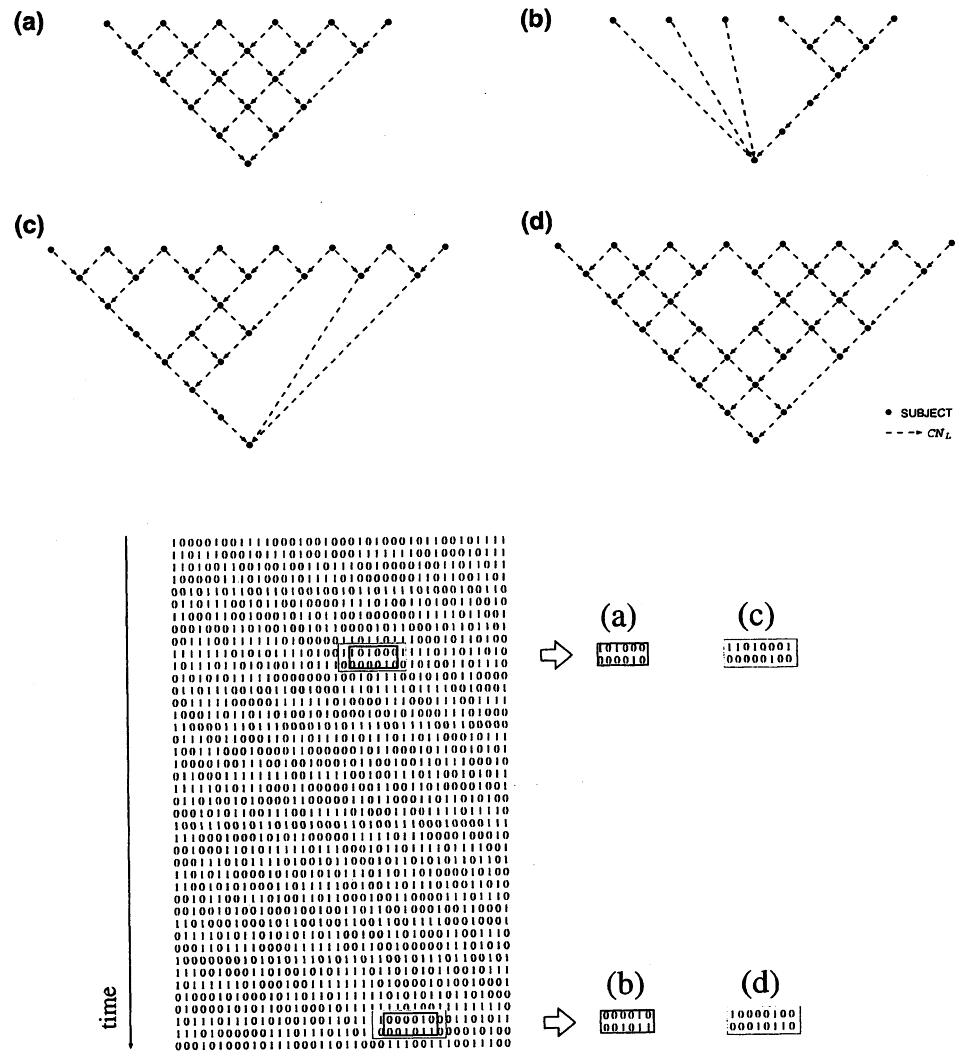

Fig. 9. The whole of necessity for four SUBJECTs. The corresponding partial state specifications of (a) and (b) specify 6 sites width regions, and the partial state specifications of (c) and (d) specify 8 sites width regions, including those of (a) and (b), respectively. The notation is the same as in Fig. 8. The lower figure shows extractions, from a space–time pattern, of the partial state specifications to corresponding the SUBJECTs of (a)–(d), respectively. The space–time pattern is the same as that shown in Fig. 5(d), with 0 and 1 representing local states.

Applying the proposed framework to the entire world enables us to describe how parts of the world with large and complex content (the whole of necessity) are distributed in the world, and how they behave in the world's dynamics. Although the discussion in this section is elementary, it provides a prospect for how the higher-order systematicity in more complex worlds will be described by our proposed framework.

5. Discussion

In this section, some related topics remaining for future investigation are mentioned, and a final conclusion is presented.

5.1. *Difficulty in covering all SUBJECTs*

Even in worlds with state-sets containing finite elements, as its size enlarges the total number of ASLPs increases combinatorially, and hence it becomes impractical to cover all of the SUBJECTs. In worlds with state-sets containing

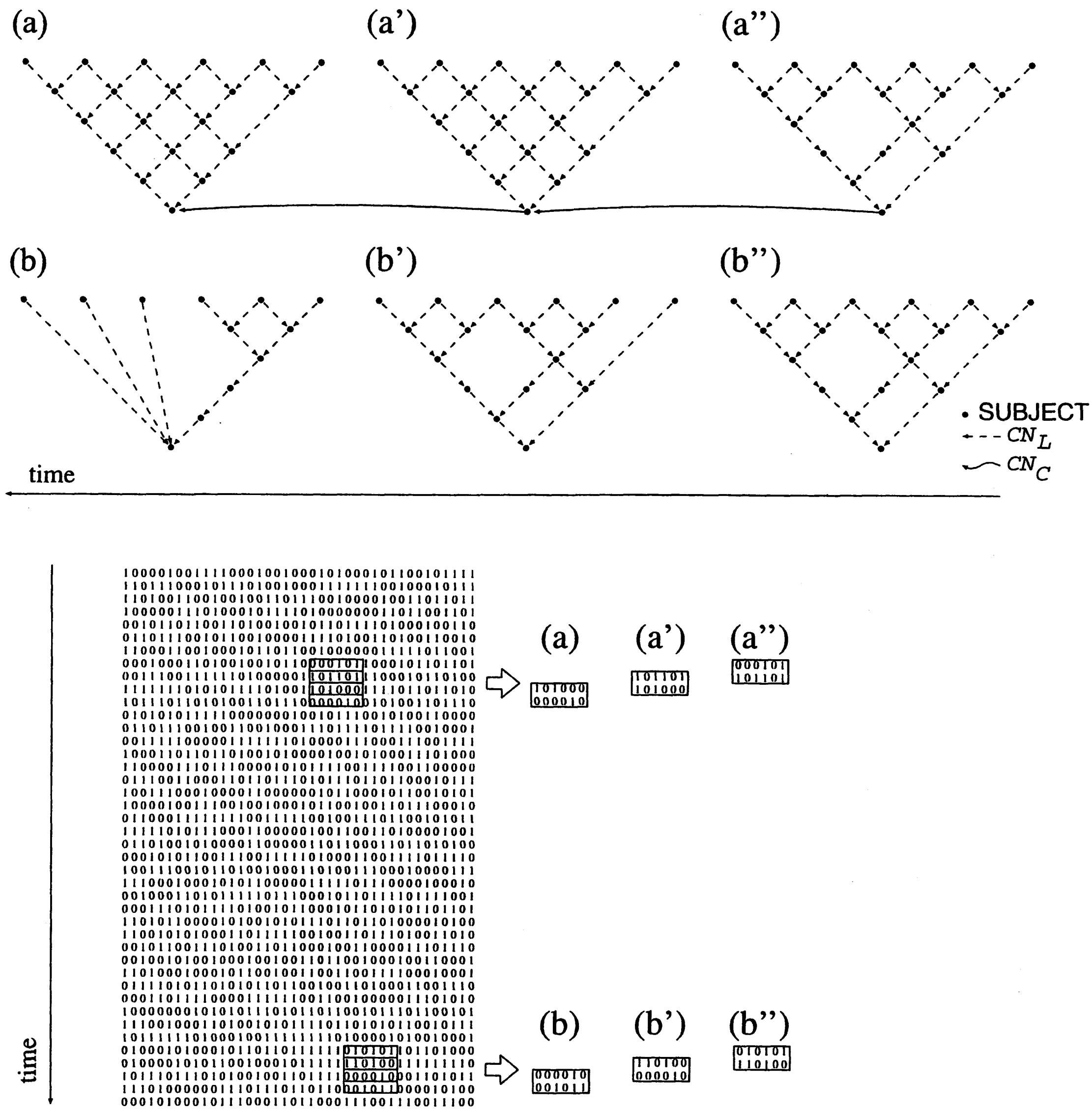

Fig. 10. The whole of necessity for six SUBJECTs, whose corresponding partial state specifications specify two 6 sites width regions at three successive pairs of times. The notation is the same as in Fig. 8. From (a″) to (a′) to (a), the SUBJECTs are connected with CN_C. From (b″) to (b′) to (b), no CN_C holds between the SUBJECTs. The lower figure shows extractions, from a space–time pattern, of the partial state specifications corresponding to the SUBJECTs of (a)–(a″) and (b)–(b″), respectively.

infinite elements, as the total of ASLPs becomes infinite, it is even basically impossible to cover all of the SUBJECTs in a procedural way, unless some highly restrictive condition is imposed at (3).

Therefore, it becomes necessary to estimate the systematicity in the world's dynamics by applying our framework to some limited set of SUBJECTs in $\mathscr{S}$. Although how a set of such "essential" SUBJECTs can be determined is a problem faced in studying each specific world, for example, the concept of generating partition in the theory of dynamical system [8] may be applicable. Let $\mathfrak{P}$ denote a generating partition of the world's state-set. First, we focus on the SUBJECTs whose corresponding ASLPs are cells of refinements, $\mathfrak{P}_n \equiv \bigvee_{i=0}^{n} \xi^{-i}\mathfrak{P}$, containing ω'^{t_0}, and study the skeletal structure of the flow of CN_C between these SUBJECTs. Then, we study how the SUBJECTs corresponding to other ASLPs are added to this skeletal structure.

Yet the general specification of generating partition is a difficult problem. Therefore, practical applications of the methods proposed for the study of chaotic systems have been restricted to low-dimensional systems (e.g., [9]). To apply the framework to larger worlds utilizing concepts in nonlinear dynamics, such as generating partition, more progress in the conventional methods will be needed.

5.2. Relation to discussions regarding the modelling of complex systems

The framework proposed in this paper is originally to describe a single world's systematicity. However it may shed some light on general discussions about the extraction of universality in natural phenomena and about "good model" to describe it [10].

A good model is often discussed in terms of its structural stability. A model being structurally stable means that the behavior of the model does not change topologically for a slight change in the model's time evolution rule. Satisfaction of this stability is usually considered to be an assurance that the model's behavior is reproducible.

On the other hand, some people claim that it is essential for the computation power or adaptability of biological (natural) systems to take advantage of a kind of instability which brings breakdown in the simulation relation [11]. Here, the simulation relation means the existence of a homomorphism which maps the description of system from one view-level to the one from another (coarser) view-level. The breakdown in this relation is considered to occur when the system's nature is not secured in a single view-level. For example, in [12], the importance of such instabilities, especially in cross-scale interactions for biological organizations, is stressed.

Corresponding with the ideas of this discussion, the question that the requirement of structural stability may be too restrictive for good models has been proposed. For example, in the context of renormalization theory, it is discussed that a good model should be structurally stable with respect to the reproducibly observable aspects, but unstable with respect to the hard-to-reproduce aspects of the actual system [13].

Including the definition of structural stability, these discussions were proposed within the conventional framework based on the concept of the state-set. It will be an interesting subject for future study to reconstruct such discussions based only on the difference of worlds described in our proposed framework.

5.3. Conclusion

In this study, we have proposed a new framework to describe the systematicity in the dynamics of deterministic worlds, substituting for the conventional framework based on the concept of the state-set. Our framework describes the world's systematicity in terms of the SUBJECTs and the connections of necessity among them, not requiring an external observer to evaluate the content of how the world actually is, as "among other possibilities, it is so." Another feature of our framework is that it equally allows for the world's states at all levels to be its primary elements, SUBJECTs, and does not presuppose an "atomic level" for the systematicity (cf. [14]).

The discussions in this paper have been limited to the effect of the differences in some basic features of worlds, on the description of the systematicity. In applying our framework to some large and complex worlds, we will study how differences in higher-order systematicity can be described. In addition, as we have focused in this paper on the redescription of worlds described by the conventional framework using our proposed framework, in the future we will explore the contrary direction, "in what world in the conventional framework would a given world's systematicity described in our proposed framework be embodied".

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